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Reducing Minimum Time for Flexible Body Small-Angle Slewing with Vibration Suppression

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Introduction

INPUT commands for slewing a flexible body in minimum time are often preshaped to suppress residual vibration after slewing. The procedure is to convolve the bang-bang command for minimum time slew for a rigid body with a three impulse sequence[1] designed to suppress the vibration of the actual, flexible body. The resulting slew extends the rigid-body minimum time by the period of vibration. A method of input shaping that does not extend the rigid-body minimum time has recently been proposed,[2] and it does so by increasing the torque level so as to reduce the rigid-body minimum time by the period of vibration. In the case of rapid slewing of a relatively soft flexible body through a small angle, it may happen that the vibration period is longer than the rigid-body minimum time, and then this method becomes inapplicable. This Note shows two exact methods to reduce the minimum time for rest-to-rest small-angle slewing in such situations. The first method is new and consists of scaling the torque following convolution, and the second method is similar to those in Refs. 3–5, except that it does not require using an optimization code. The methods are robust and illustrated with systems having one or two vibration modes to suppress. Application of the methods may be as diverse as in disk drive positioning and spacecraft precision slewing.

Scaling of Convolved Torque

Consider first a system, with one vibration mode to suppress, represented by two rigid rotors of inertia J_1, J_2 connected by a massless elastic shaft of stiffness K. Let $J_1 = 1.0$, $J_2 = 0.1$ kg-m^2, and $K = 4.3426$ N-m/rad, so that the time period of vibration is 10 s. Minimum time bang-bang control for a rigid body is described by the torque time function,

$$\begin{aligned} \tau_b &= T_{\max}, && 0 \le t \le t_{\min}/2 \\ &= -T_{\max}, && t_{\min}/2 < t \le t_{\min} \\ &= 0, && t > t_{\min} \end{aligned} \tag{1}$$

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where the minimum time for slewing to an angle θ_f is given by

$$t_{\min} = 2\sqrt{J_t\theta_f/T_{\max}} \tag{2}$$

For $\theta_f = 1$ deg and the total rigid-body inertia in this case $J_t = 1.1$, the minimum time is 8 s for a maximum torque $T_{\max} = 0.0012$ N-m. This is a case where the rigid-body slewing time is less than the vibration period. Input shaping to suppress the vibration that would be generated by the bang-bang torque consists in doing a convolution of the time function of Eq. (1) with the three impulse sequence time function given in Table 1, where the function values are zero except at the indicated time points.

This convolution is symbolically represented as follows:

$$T = \tau_b^* F_1 \tag{3}$$

The top plot in Fig. 1 shows the result of the convolved torque in this numerical case. The key point to note is that the maximum torque used is less than the available maximum torque. In fact, it can be shown that the torque used is less than the available maximum whenever the period of vibration to be suppressed is greater than half the minimum time for a rigid body. This suggests the idea that a new solution can be sought by scaling up the original maximum torque, thereby reducing the rigid-body minimum time so that after convolution with the three impulse sequence the actual required maximum torque equals the available maximum torque. The factor by which the maximum torque has to be scaled up depends on the result of the convolution. The torque factor (available torque/maximum torque after convolution) is two in this case, and the result of the scaled and convolved torque is shown as the middle plot in Fig. 1. The corresponding time plots of the slew angle in the two cases of unscaled and scaled torques are shown in the bottom plot of Fig. 1. It is seen that scaling the torque has reduced the minimum time.

Although simple linear scaling works if only one mode shaping is involved, this does not work for multimode vibration suppression. To see this, consider a three-degree-of-freedom system of three rigid bodies connected by two mass-less elastic beams. The bodies are of moments of inertia J_1, J_2, and J_3, and the beams are of bending stiffnesses K_1 and K_2. The system is undergoing planar slewing

Table 1 Function F_i representing three impulse sequence for vibration of period T_i

t	F_i
0	0.25
$T_i/2$	0.5
T_i	0.25

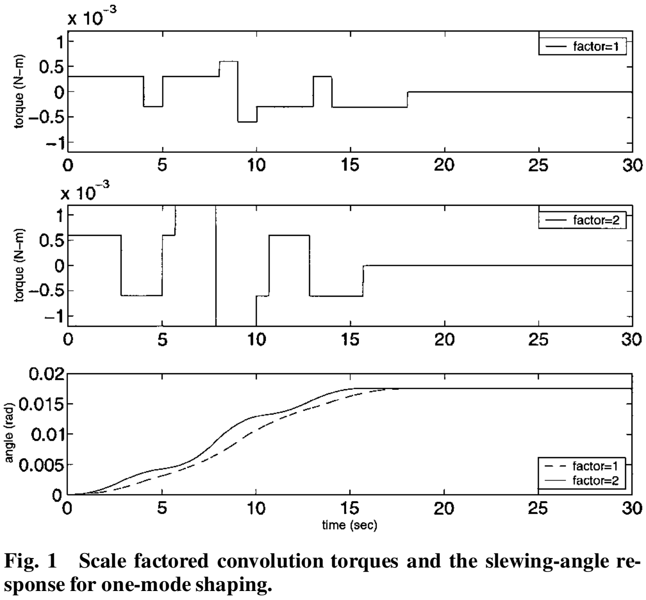

Fig. 1 Scale factored convolution torques and the slewing-angle response for one-mode shaping.

with torque T acting on the body of inertia J_1. The equations of motion of the flexible body system in modal form are

$$J_T\ddot{q}_r = T, \qquad \ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 = \phi_1 T$$
$$\ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2 q_2 = \phi_2 T \tag{4}$$

and the slew angle θ_1 of the first body is

$$\theta_1 = q_r + \phi_1 q_1 + \phi_2 q_2 \tag{5}$$

Here J_T is the total inertia of the system; q_r is the rigid-body mode coordinate; q_1 and q_2 are the two elastic mode coordinates; ζ_1, ζ_2 are the modal damping factors; and ω_1, ω_2 are the natural frequencies obtained from the characteristic equation:

$$\omega^4 - \left[\frac{k_1}{J_1} + \frac{k_2}{J_2}\left(1 + \frac{k_1}{k_2} + \frac{J_2}{J_3}\right)\right]\omega^2 + \frac{k_1}{J_1}\frac{k_2}{J_2}\frac{J_T}{J_3} = 0 \tag{6}$$

For a specific example, numerical values of $J_1 = J_2 = J_3 = 15441$ kg-m^2 and values of $K_1 = 2661.6$ and $K_2 = 30498.8$ in N-m/rad are chosen to yield natural frequencies of 0.08 Hz and 0.32 Hz. Using a total inertia of $J_1 + J_2 + J_3$ and assuming a maximum available torque of 0.28 N, the minimum time for slewing through a small angle of 20 arc-seconds can be computed from Eq. (2) as 19.62 s. With the standard method of convolution of the bang-bang command with the three impulse sequence for two modes, the total slewing time is the sum of the minimum time and the two vibration periods, giving a total slewing time of 35.245 s in this case. Denoting by F_1 and F_2 the two sets of three impulse functions of Table 1 corresponding to vibrations of periods T_1 and T_2, the combined convolution of the functions τ_b and these two functions can be symbolically represented as

$$T = \tau_b^*\left(F_1^* F_2\right) \tag{7}$$

The top plot in Fig. 2 shows the result of doing this convolution as per Eq. (7) using the parameters of the two-mode illustrative problem. It is seen that convolved torque produced by Eq. (7) is far more complex in shape than the simple bang-bang torque of Eq. (1). The torque goes to zero at 35.245 s as expected, but the interesting fact is that the maximum value of the convolved torque, 0.21 N-m, is less than the available maximum $T_{\max} = 0.28$ N. Again, the convolved torque command does not utilize the available maximum torque. Using the factor of 1.333, which is the ratio of the available torque to the maximum value of the convolved torque, one obtains the middle plot in Fig. 2. It is seen that the convolved maximum torque does not linearly scale to reach the available maximum. This suggests the use of scaling up the nominal bang-bang torque by an unknown factor α so that after convolution maximum torque equals the available torque:

$$\begin{aligned}\tau_b(\alpha) &= \alpha T_{\max}, & 0 \le t \le t_{\min}\big/\left(2\sqrt{\alpha}\right)\\ &= -\alpha T_{\max}, & t_{\min}\big/\left(2\sqrt{\alpha}\right) < t \le t_{\min}\sqrt{\alpha}\\ &= 0, & t > t_{\min}\big/\sqrt{\alpha}\end{aligned} \tag{8}$$

where $t_{\min}$ is given by Eq. (2) as before. Now one solves the equation,

$$\max\left[\tau_b(\alpha)^*\left(F_1^* F_2\right)\right] = T_{\max} \tag{9}$$

One has to solve the nonlinear equation, Eq. (9), in principle only; in practice this is done by trial and error use of Eq. (9) with a few iterations. Figure 2 shows progressively the behavior of the convolved torque, with the factor $\alpha = 2$ realizing the case of required maximum torque matching available torque. The slewing maneuver is over in 29.82 s. The modest 15% reduction in time from the unscaled shaped input, corresponding to factor = 1 in Fig. 2, is caused by the fact that the original convolution uses the maximum torque only for a very small fraction of the total time. Reducing the minimum time even further for multimode vibration suppression is the objective of the next section.

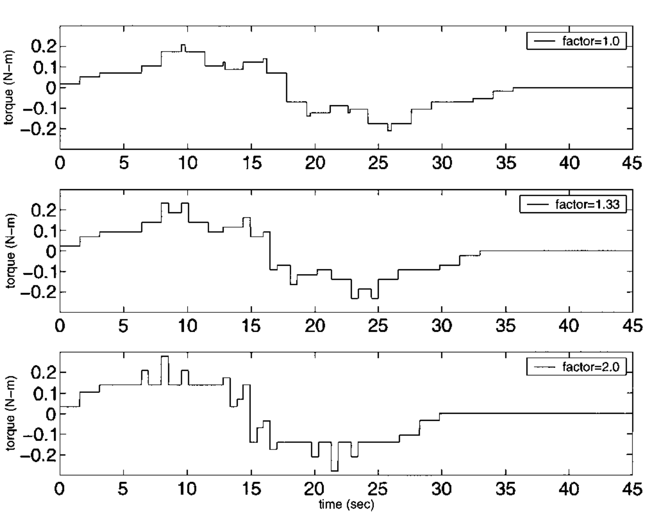

Fig. 2 Scale factored minimum time torque convolved with two-mode shaper.

Bang-Bang Control

It is well known from Pontryagin's maximum principle that minimum time control, with control T appearing linearly in the dynamical equations, is a bang-bang torque in time, and the unknowns are the number and instants of switching times. A multiple bang-bang pulse can be looked upon as a convolution of a step function with multiple impulses of strength A_i occurring at times t_i. For a normalized step of unity, this specifies the use of the Table 2 of timed impulses.

Taking t_1 to be zero, without loss of generality, this presents n unknown switching times. For a solution to exist, n must be greater than or equal to the number of governing equations.

The conditions of zero residual vibration in the mode of frequency ω_j ($j = 1, 2$) with the use of the impulse-time data of Table 2 yield the following equations:

$$2\sum_{i=2}^{n}(-1)^{i-1}\sin\omega_j t_i + \sin\omega_j t_{n+1} = 0, \ldots, j = 1, 2 \tag{10}$$

$$1 + 2\sum_{i=2}^{n}(-1)^{i-1}\cos\omega_j t_i + \cos\omega_j t_{n+1} = 0, \ldots, j = 1, 2 \tag{11}$$

Insensitivity to frequency modeling error[1] gives rise to

$$2\sum_{i=2}^{n}(-1)^{i-1}t_i\sin\omega_j t_i + t_{n+1}\sin\omega_j t_{n+1} = 0, \ldots, j = 1, 2 \tag{12}$$

$$2\sum_{i=2}^{n}(-1)^{i-1}t_i\cos\omega_j t_i + t_{n+1}\cos\omega_j t_{n+1} = 0, \ldots, j = 1, 2 \tag{13}$$

Terminal conditions for slewing to a desired angle θ_f for colocated measurement are

$$\theta_1(t_f) = \theta_f \tag{14}$$

$$\dot{\theta}_1(t_f) = 0 \tag{15}$$

Table 2 Impulse strength and switching times for bang-bang control

t_i	A_i
t_1	1
t_2	−2
t_3	2
t_4	−2
—	—
t_n	−2
t_{n+1}	1

Table 3 Switching times and impulse for bang-bang control with two mode suppression

Time	Amplitude
0	1
3.8689	−2
4.6629	2
6.3546	−2
6.9040	2
11.2638	−2
15.8141	2
16.9831	−2
21.1091	2
21.6511	−2
23.2629	1

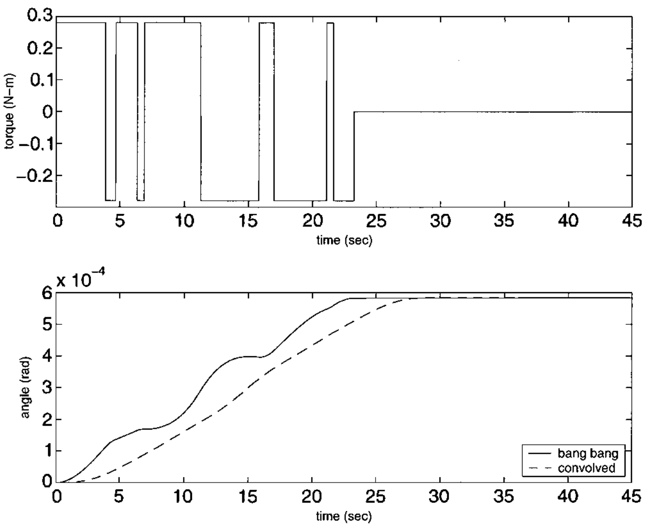

Fig. 3 Bang-bang torque and slew-angle responses caused by it and the convolved torque of Fig. 2 with factor = 2.

Equations (10–15) constitute 10 equations in 10 unknowns, which can be exactly solved No optimization problem needs to be solved. Unlike Refs. 3–5, which obtained solutions that minimize the final time by numerical optimization, a nonlinear equation solver[6] was used in this paper to solve Eqs. (10–15). Minimality of the solution is realized here by creeping up on the time axis for a first, feasible solution.

Both nonlinear equations solvers and optimization codes require good initial guesses to converge locally. The solution process consists in using exact solutions for the linear differential equations, Eqs. (4), to set up Eqs. (14) and (15) via Eq. (5). The switching times, obtained from a symbol manipulation code[7] based on the theory in Ref. 6 for the numerical parameters of this example, are in Table 3.

The top plot of Fig. 3 shows the torque pulse time history obtained by convolution of a step function with the impulse train of Table 3. The bottom plot in Fig. 3 compares the slewing-angle response given by the scaled convolved torque and the bang-bang torque. It is seen that bang-bang torque gives the greater reduction in the time of slew than the scaled convolution torque, whereas the scaled convolution torque induces less transient vibration.

Robustness

Robustness of both the scaled convolution and the bang-bang solutions to errors in modeling frequency and damping rests on the basic robustness property typified by Eqs. (12) and (13) of the input shaping process.[1] Figure 4 demonstrates for the two methods the results of applying a shaped torque designed for a nominal system to a system with different frequency and damping. Referring to Eq. (4), the natural frequency in two modes were increased by 10%

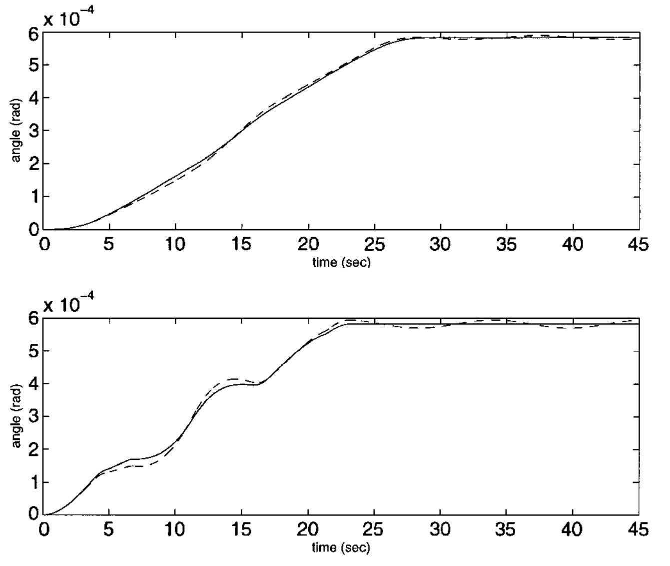

Fig. 4 Robustness results for scaled convolution solution and bang-bang solution (——, nominal; - - - -, application to model with frequency raised 10%; and ·····, nominal plus 1% damping, indistinguishable from nominal solutions).

from the nominal and zero damping is assumed; to examine damping robustness, damping was taken to be 1% with nominal frequencies.

The results show frequency robustness of the solutions, with the scaled convolution method more robust than the bang-bang solution. In both cases the effect of adding damping is hardly distinguishable from the nominal solutions. Apparently, inherent modal damping does not change the results.

Conclusions

Two exact methods are given for reducing the minimum time for small-angle slewing of a flexible body when the period of vibration to be suppressed is greater than the minimum time for rigid-body slewing. One method requires only a scaling of the torque, the scale factor depending on the maximum level of the torque used in the process of convolution of the rigid-body minimum time torque with the three impulse sequence. The method is simple and requires only scaling of a convolution. For multimode vibration suppression a few iterations are required for determining the scale factor that realizes the use of the maximum available torque. The second method requires solving as many nonlinear equations as the number of switching times. This method is applicable irrespective of the separation of slew time and the period of vibration. The bang-bang solution gives the faster slewing time, whereas the scaled convolution method induces less transient vibration suppression. Robustness of both methods is shown with an example of two-mode vibration suppression following slewing.

Acknowledgment

The author thanks his colleague, Nelson Pedreiro, for suggesting this problem.

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Optimal Control Bolza and Transformed Mayer Problems with Feedback Linearized State Equations

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I. Introduction

Minimum time solution of classes of linear and nonlinear dynamic systems with bounded inputs is well known to be bang–bang, that is, the optimal inputs switch between minimum and maximum values.[1−3] Structures of their solution are well documented in the literature. Mayer problems for classes of feedback linearizable nonlinear systems with mixed input and state constraints also have special structures: At least one constraint is always on the boundary.[4] This result is a generalization of the classical bang–bang result for linear systems.

In this paper, we consider Bolza optimal control problems where the dynamic systems are given by

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathcal{R}^n$ are the states, $u(t) \in \mathcal{R}^m$ are the controls, and the matrix $g(x)$ consists of smooth vector fields $g_1(x), \ldots, g_m(x)$. The objective is to

$$\text{minimize } J = \Phi[x(t_f), t_f] + \int_{t_0}^{t_f} F(x, u)\,\mathrm{d}t \tag{2}$$

where $t \in [t_0, t_f]$ denotes the time. The scalar functions $\Phi[x(t_f), t_f]$ and $F(x, u)$ have continuous first and second partial derivatives with respect to their arguments. The inequality constraints are $s(x, u) \leq 0$ and $c(x) \leq 0$. The terminal states satisfy $\psi[x(t_0), x(t_f)] = 0$ with $\partial\Phi/\partial t_f \neq 0$.

A Mayer optimal control problem does not have an explicit integral in its cost: This is a major difference between Bolza and Mayer optimal control problems. As is well known, every Bolza problem can be converted to a Mayer problem through the following two steps: An auxiliary state x_{n+1} is added to Eq. (1) with the dynamics $\dot{x}_{n+1} = F(x, u)$, thereby, modifying the state equations to

$$\dot{x} = f(x) + g(x)u, \qquad \dot{x}_{n+1} = F(x, u) \tag{3}$$

The cost in Eq. (2) can now be rewritten as

$$\text{minimize } J = \Phi[x(t_f), t_f] + x_{n+1}(t_f) \tag{4}$$

with $x_{n+1}(t_0) = 0$. In this new form of the problem, the inequality constraints and boundary constraints do not change.

As outlined in Ref. 4, special structures are present in the optimal solution of a Mayer problem if the governing system dynamics are feedback linearizable, that is, diffeomorphic to chains of integrators. Such structures will also be present in the solution of a Bolza problem if the set of Eqs. (3) is feedback linearizable. This motivates the statement of the problem discussed in this Note: given the state Eqs. (1) and the integrand $F(x, u)$ in Eq. (2), what forms of $F(x, u)$ will make the augmented state Eqs. (3) feedback linearizable?

The organization of this Note is as follows: Section II derives the necessary conditions for Eqs. (3) to be feedback linearizable. Consistent with these necessary conditions, linear and quadratic forms of $F(x, u)$ are studied in detail in Secs. III and IV, respectively. The results of this paper are illustrated by an example in Sec. V.

II. Conditions on $F(x, u)$

A. Background Results

Definition 1: A system $\dot{x} = f(x, u)$, $x \in R^n$, $u \in R^m$ is said to be differentially flat if there exists $y \in R^m$ dependent on x, u, and their derivatives up to a finite order such that x and u can be written as functions of y and its derivatives up to a finite order.

Flat systems are dynamic feedback linearizable.[5,6] Necessary and sufficient conditions do not exist that guarantee a general system to be flat. However, there are results for special classes of systems. For example, flatness is equivalent to controllability for linear systems. Other systems that are known to be flat are static feedback linearizable systems, which are characterized as follows[7,8]:

Theorem 1: A control affine system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

is static feedback linearizable if and only if the following distributions are constant rank and involutive:

$$D_0 = \langle g_1, \ldots, g_m \rangle$$

$$D_i = \left\langle D_{i-1}, ad_f^i g_1, \ldots, ad_f^i g_m \right\rangle, \qquad \forall i = 1, \ldots, n-1$$

and D_{n-1} spans R^n.

Proposition 1: A single-input system is dynamic feedback linearizable if and only if it is static feedback linearizable. The proof can be found in several papers and textbooks, for example, Refs. 6 and 9.

Proposition 2: A necessary condition for a system

$$\dot{x} = f(x, u), \qquad u \in R^m \tag{5}$$

$$\dot{x}_{n+1} = F(x, u) \tag{6}$$

to be dynamic feedback linearizable is that Eq. (5) be dynamic feedback linearizable.

Proof: A dynamic compensator for Eq. (5) is

$$\dot{z} = a(x, z, v), \qquad u = b(x, z, v) \tag{7}$$

Therefore, if

$$z = x_{n+1}, \qquad F(x, u) = a(x, z, v), \qquad u = v$$

we can view Eq. (6) to be a dynamic compensator for Eq. (5). Then, the dynamic feedback linearizability of Eqs. (5) and (6) implies dynamic feedback linearizability of Eq. (5). That is, Eq. (5) is dynamic feedback linearizable. □

Corollary 1: If Eq. (5) is restricted to a single input and the overall system is dynamic feedback linearizable, then Eq. (5) is static feedback linearizable.

Proof: It is clear from the preceding proposition and equivalence between static and dynamic feedback linearization for single-input systems.

Corollary 2: If Eqs. (5) and (6) are linear and the overall system is controllable, then Eq. (5) is controllable.

Received 30 November 2000; revision received 12 June 2001; accepted for publication 15 June 2001.

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